Fast and simple constant-time hashing to the BLS12-381 elliptic curve

(and other curves, too!)

Riad S. Wahby, Dan Boneh

Stanford

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Why the BLS12-381 pairing-friendly elliptic curve?

- Widely used curve for ≈120-bit security level
  - ☞ Will (probably) be an IETF standard soon

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   ☞ Open-source impls in C, Rust, Python, ...

Roadmap

1. Hash functions to elliptic curves
2. Optimizing the map of [BCIMRT10]
3. Evaluation results
4. IETF standardization efforts

Notation

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1. Seed a PRG with the input
2. Extract a $2\log p$-bit integer
3. Reduce mod p

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$E(\mathbb{F}_p)$ is the elliptic curve group with identity $\mathcal{O}$ and points $\{(x,y) : x,y \in \mathbb{F}_p, y^2 = x^3 + ax + b\}$
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BLS12-381 defines $\mathbb{G}_1 \subset E_1(\mathbb{F}_p)$, $\mathbb{G}_2 \subset E_2(\mathbb{F}_{p^2})$, $\mathbb{G}_T \subset \mathbb{F}_{p^{12}}^\times$, and $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$ s.t.

$$e([\alpha]P_1, [\beta]P_2) = e(P_1, P_2)^{\alpha \cdot \beta} \qquad \alpha, \beta \in \mathbb{F}_q$$

Attempt #1: random scalar

For some distinguished point $\hat{P} \in \mathbb{G}$,

$\mathsf{HashToCurve}_{\mathsf{RS}}(\mathsf{msg})$:

$x \leftarrow H_q(\mathsf{msg})$

return $[x]\hat{P}$

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Informally: need a point with unknown discrete log

☞ known dlog breaks security of most protocols (e.g., BLS signatures)

BLS signatures

For $H : \{0,1\}^\star \to \mathbb{G}_1,\ \hat{Q} \in \mathbb{G}_2$:

KeyGen() $\to (pk, sk)$:

$r \xleftarrow{\mathrm{R}} \mathbb{Z}_q$; return $([r]\hat{Q}, r)$

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Verify$(pk, \text{msg}, \text{sig}) \to \{\text{True}, \text{False}\}$:

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☞ Trivial existential forgery:

$$\text{Sign}(sk, \text{msg}_2) = \left[H_q(\text{msg}_2) \cdot H_q(\text{msg}_1)^{-1}\right]\text{sig}_1$$

Attempt #2: hash and check

$\mathsf{HashToCurve}_{\mathsf{H\&C}}(\mathsf{msg})$:

$\mathsf{ctr} \leftarrow 0$

$y \leftarrow \bot$

while $y = \bot$:

$x \leftarrow H_p(\mathsf{ctr} \,||\, \mathsf{msg})$

$\mathsf{ctr} \leftarrow \mathsf{ctr} + 1$

$ySq \leftarrow x^3 + ax + b$

$y \leftarrow \mathsf{sqrt}(ySq)$ // $\bot$ if ySq is non-square

$P \leftarrow (x, y)$

return $[h]P$ // map to $\mathbb{G}$ via cofactor mul

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Not constant time; "bad" inputs are common.

Attempt #2: hash and check

HashToCurve$_{\mathsf{H\&C}}$(msg):

ctr $\leftarrow 0$

$y \leftarrow \perp$

while $y = \perp$:

  $x \leftarrow H_p(\text{ctr} \,||\, \text{msg})$

  ctr $\leftarrow$ ctr $+ 1$

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✗ Loop a fixed number of times?
Slow; well-meaning “optimization” breaks CT.

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$M : \mathbb{F}_p \to E(\mathbb{F}_p)$, where $E : y^2 = x^3 + ax + b$ and $p > 5$:

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SWU	[Ulas07]	✗ $p \equiv 3 \bmod 4, ab \neq 0$	3 exp
	[Icart09]	✗ $p \equiv 2 \bmod 3$	1 exp
S-SWU	[BCIMRT10]	✗ $p \equiv 3 \bmod 4, ab \neq 0$	2 exp
Elligator	[BHKL13]	✗ $b \neq 0, 2 \mid \#E(\mathbb{F}_p)$	1 exp
This work		✗ $ab \neq 0$	1 exp
		✓ none	1^+ exp

BLS12-381: $p \equiv 1 \bmod 3, \quad a = 0, \quad 2 \nmid \#E(\mathbb{F}_p)$

[SS04,Ska05,FSV09,FT10a,FT10b,KLR10,CK11,Far11,FT12,FJT13,BLMP19...]

The Shallue–van de Woestijne map [SW06] (high level)

$$E : y^2 = f(x) = x^3 + ax + b$$

Idea #1 (Skałba): For $X_1, X_2, X_3, X_4 \neq 0$, let

$$V(\mathbb{F}_p) : f(X_1) \cdot f(X_2) \cdot f(X_3) = X_4^2$$

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☞ One of $f(X_i), i \in \{1, 2, 3\}$ must be square

$\Rightarrow$ that X_i must be an x-coordinate on $E(\mathbb{F}_p)$

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☞ constant-time cost dominated by 3 exps
(recall: Legendre symbol in $\mathbb{F}_p$ ops is 1 exp)

Hash functions from deterministic maps

Compose H_p and M in a natural way:

$\mathsf{HashToCurve}_{\mathsf{NU}}(\mathsf{msg})$:

$t \leftarrow H_p(\mathsf{msg})$ // $\{0,1\}^\star \mapsto \mathbb{F}_p$

$P \leftarrow M(t)$ // $\mathbb{F}_p \mapsto E(\mathbb{F}_p)$

return $[h]P$ // $E(\mathbb{F}_p) \mapsto \mathbb{G}$

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☞ Can use a faster method for cofactor clearing:

- via endomorphisms [GLV01,SBCDK09,FKR11,BP18]
- via subgroup structure [S19 (see WB19, §5)]

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This *could* be OK—but what if we need uniformity?

Uniform hashing from deterministic maps [BCIMRT10]

For some distinguished point $\hat{P} \in \mathbb{G}$:

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$P_1 \leftarrow M(H_p(\mathsf{msg}))$

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Idea [BCIMRT10,FFSTV13]:

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☞ Indifferentiable from RO if M is *well distributed*

✓ All of the M we've seen are well distributed.

Roadmap

1. Hash functions to elliptic curves
2. Optimizing the map of [BCIMRT10]
3. Evaluation results
4. IETF standardization efforts

The Simplified SWU map [BCIMRT10]

$$E : y^2 = f(x) = x^3 + ax + b, \qquad ab \neq 0$$

Idea: pick x s.t. $f(ux) = u^3 f(x)$.

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$$\therefore \qquad x = -\frac{b}{a}\left(1 + \frac{1}{u^2 + u}\right)$$

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☞ If $p \equiv 3 \bmod 4$, $u = -t^2$ is non-square, so:

$$X_0(t) \triangleq -\frac{b}{a}\left(1 + \frac{1}{t^4 - t^2}\right) \qquad X_1(t) \triangleq -t^2 X_0(t)$$

Evaluating the S-SWU map

$$\text{S-SWU}(t) \triangleq \begin{cases} (X_0(t), \sqrt{f(X_0(t))}) & \text{if } f(X_0(t)) \text{ is square} \\ (X_1(t), \sqrt{f(X_1(t))}) & \text{otherwise} \end{cases}$$

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Requires two exponentiations! Can we do better?

Eliminating an exponentiation

Recall: $f(x_1) = -t^6 f(x_0)$. So:

$$f(x_1)^{\frac{p+1}{4}} = \left(-t^6 f(x_0)\right)^{\frac{p+1}{4}}$$

Eliminating an exponentiation

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Legendre symbol!

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$$= -f(x_0) \qquad \text{if } f(x_0) \text{ is non-square}$$

✓ $f(x_0)^{\frac{p+1}{4}}$ is $\sqrt{-f(x_0)}$ when $f(x_0)$ is non-square!

Evaluating the S-SWU map—faster!

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✓ Straightforward to generalize to $p \equiv 1 \bmod 4$.

Generalizing: the $p \equiv 5 \bmod 8$ case

-1 is square in $\mathbb{F}_p \Rightarrow$ need $u = \xi t^2$ for ξ nonsquare.

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☞ ξ is fixed, so we can preompute $\left(\xi^3\right)^{\frac{p+3}{8}}$

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Issue: S-SWU still does not work with $ab = 0$.

☞ Rules out pairing-friendly curves [BLS03,BN06,...]

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Idea: map to a curve E' having $ab \neq 0$ and an efficiently-computable homomorphism to E.

Supporting the $ab = 0$ case

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☞ Rules out pairing-friendly curves [BLS03,BN06,...]

Idea: map to a curve E' having $ab \neq 0$ and an efficiently-computable homomorphism to E.

Specifically: Find $E'(\mathbb{F}_p)$ d-isogenous to E, d small.
☞ Defines a degree $\approx d$ rational map $E'(\mathbb{F}_p) \to E(\mathbb{F}_p)$

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Then: S-SWU to $E'(\mathbb{F}_p)$, isogeny map to $E(\mathbb{F}_p)$.
✓ Preserves well-distributedness of S-SWU.

Roadmap

1. Hash functions to elliptic curves
2. Optimizing the map of [BCIMRT10]
3. Evaluation results
4. IETF standardization efforts

Implementation, baselines, setup, method

BLS12-381 defines $\mathbb{G}_1 \subset E_1(\mathbb{F}_p)$ and $\mathbb{G}_2 \subset E_2(\mathbb{F}_{p^2})$.

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For $\mathbb{G}_1$ and $\mathbb{G}_2$, we implement:

- Maps: hash-and-check; [SW06]; this work
- Styles: full bigint; field ops only, non-CT and CT
- Hashes: non-uniform; uniform

In total: 34 hash variants, 3520 lines of C.

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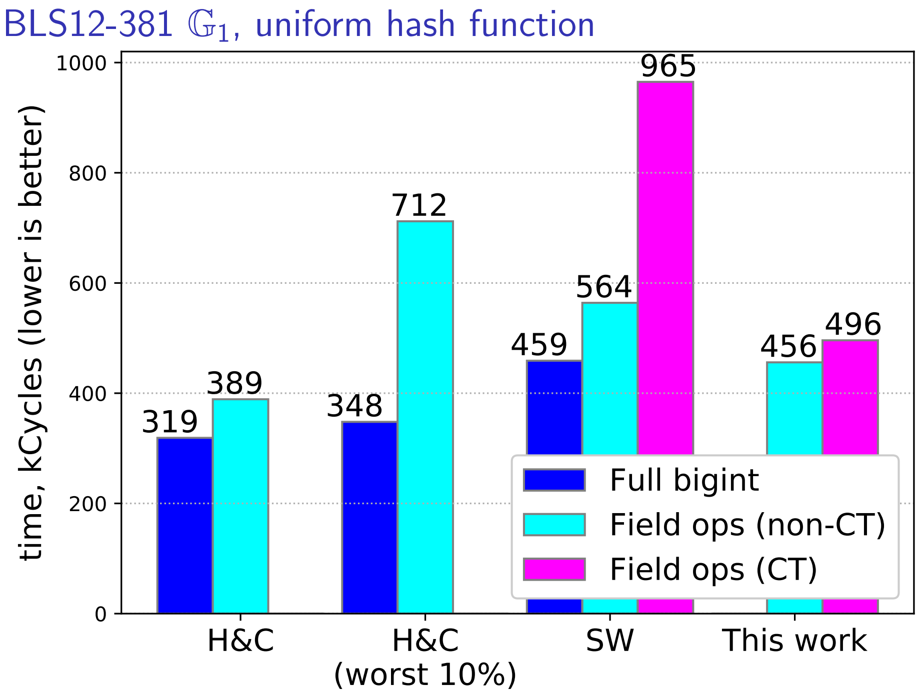
Method: run each hash 10^6 times; record #cycles.



BLS12-381 $\mathbb{G}_1$, uniform hash function
time, kCycles (lower is better)
0
200
400
600
800
1000
319
389
348
712
459
564
965
456
496
H&C
H&C (worst 10%)
SW
This work
Full bigint
Field ops (non-CT)
Field ops (CT)

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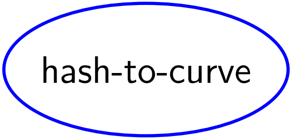
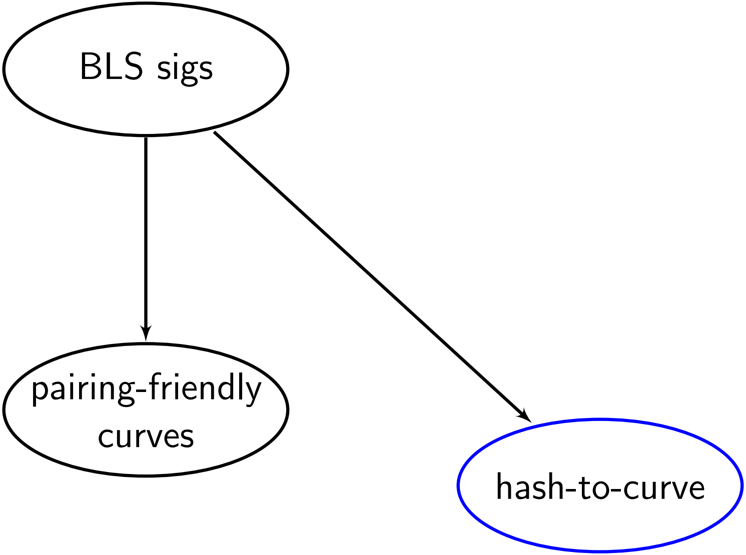

hash-to-curve


BLS sigs
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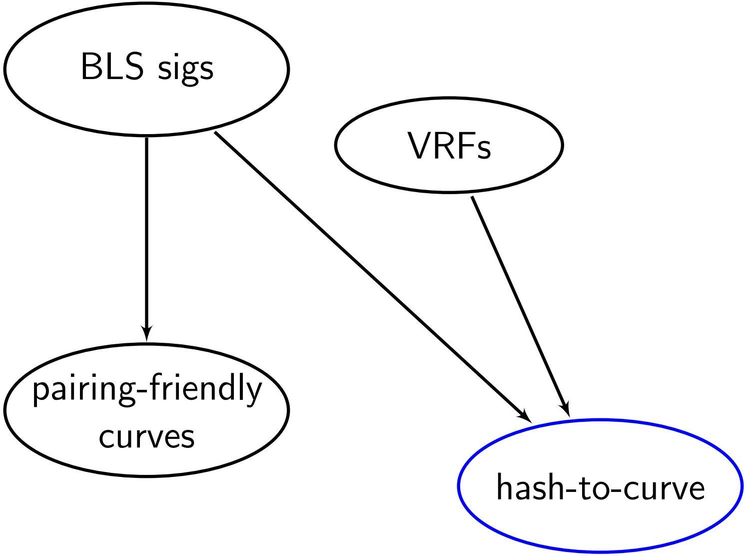
BLS sigs
VRFs
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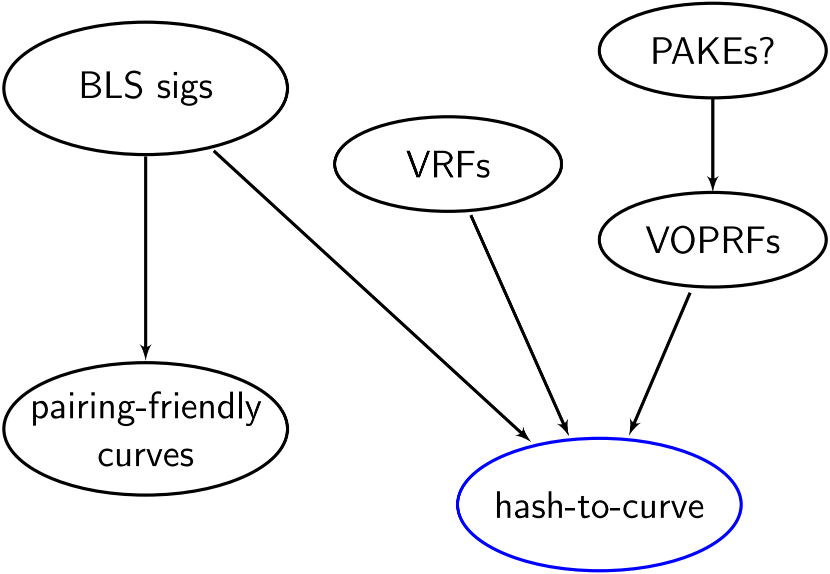
BLS sigs
PAKEs?
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VOPRFs
pairing-friendly curves
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Which maps should the IETF standardize?

$M : \mathbb{F}_p \to E(\mathbb{F}_p)$, where $E : y^2 = x^3 + ax + b$ and $p > 5$:

Map M		**Restrictions**	**Cost**
	[BF01]	$p \equiv 2 \bmod 3$, $a = 0$	1 exp
	[SW06]	none	3 exp
SWU	[Ulas07]	$p \equiv 3 \bmod 4$, $ab \neq 0$	3 exp
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[SS04,Ska05,FSV09,FT10a,FT10b,KLR10,CK11,Far11,FT12,FJT13,BLMP19...]

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☞ What about supersingular maps [BF01,BLMP19]?

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Contributions:

✓ Optimizations to the map of [BCIMRT10]

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`https://bls-hash.crypto.fyi`

`https://github.com/kwantam/bls12-381_hash`

`https://github.com/cfrg/draft-irtf-cfrg-hash-to-curve`

`rsw@cs.stanford.edu`